On Estimates in Hardy Spaces for the Stokes Flow in a Half Space (Harmonic Analysis and Nonlinear Partial Differential Equations)

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We consider the Stokes equation

$$
\begin{aligned}
&u_t - \Delta u + \nabla p = 0, \mathrm{div} u = 0 \text{ in } \Omega \times (0,\infty), \\
&u = u_0 \text{ at } t = 0, \\
&u = 0 \text{ on } \partial\Omega \times (0,\infty)
\end{aligned} \tag{1}
$$

in a domain Ω in $\mathbb{R}^n (n \geq 2)$ with smooth boundary. Here $u = (u^1, \ldots, u^n)$ is unknown velocity field and p is unknown pressure field. Initial data u_0 is assumed to satisfy a *compatibility condition* : $\mathrm{div} u_0 = 0$ in Ω and the normal component of u_0 equals zero on $\partial\Omega$. This system is a typical parabolic equation and it has several properties resembling to the heat equation.

If $\Omega = \mathbb{R}^n$, u is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. For example regularity-decay estimate

$$
\|\nabla u(t)\|_p \leq C t^{-1/2} \|u_0\|_p \text{ for } t > 0 \tag{2}
$$

holds for all $1 \leq p \leq \infty$ with C independent of t and u_0, where $\|f(t)\|_p := \left(\int_\Omega |f(t,x)|^p dx\right)^{1/p}$ and ∇ denotes the gradient in space variables. If $p = 2$, the estimate (2) is still valid for any domain. Indeed, since the Stokes operator A is self–adjoint and nonnegative, the operator A generates an analytic semigroup e^{-tA}. This yields

$$
\|A^{1/2} e^{-tA} u_0\|_2 \leq C t^{-1/2} \|u_0\|_2.
$$

Since $u = e^{-tA} u_0$ and $\|A^{1/2} u\|_2 = \|\nabla u\|_2$, (2) follows for $p = 2$.(See Borchers and Miyakawa [3] for applications.) For $1 < p < \infty$, (2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [13]). The estimate (2) is also valid for exterior domain with $n \geq 3$, with extra restriction $1 < p < n$.(See Borchers and Miyakawa [2], Giga and Sohr [8], Iwashita [10].)

However, there was no result for $p = 1$ or $p = \infty$ where the boundary of Ω is not empty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 type spaces, because it involves the singular integral operator such as Riesz operators. Nevertheless, we prove (2) for $p = 1$ where Ω is a half space $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n); x_n > 0\}$.

Theorem 1. *Let u be the solution of the Stokes equation (1) in $\Omega = \mathbb{R}^n_+$ with initial data $u_0 \in L^1(\mathbb{R}^n)$, which satisfies the compatibility condition. Then there is a constant C independent of u_0 such that*

$$\|\nabla u(t)\|_1 \leq Ct^{-1/2}\|u_0\|_1 \tag{3}$$

for all $t > 0$.

This is rather surprising since we do not expect $\|u(t)\|_1 \leq C\|u_0\|_1$ for $\Omega = \mathbb{R}^n_+$. Actually, the estimate (3) follows from a stronger estimate:

Theorem 2. *Under the same hypothesis of the Theorem 1, there is a constant C' independent of u_0 such that*

$$\|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \leq C't^{-1/2}\|u_0\|_1 \tag{4}$$

for all $t > 0$.

Here

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n_+)} = \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)}; \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n),\ \tilde{f}|_{\mathbb{R}^n_+} \equiv f\},$$

where $\mathcal{H}^1(\mathbb{R}^n)$ is the Hardy space in $\mathbb{R}^n$ with a norm

$$\|f\|_{\mathcal{H}^1} = \|f^*\|_{L^1(\mathbb{R}^n)} = \|\sup_{s>0}|f * G_s|\|_{L^1(\mathbb{R}^n)}.$$

Here G_s is the Gauss kernel.

To show (4), we recall the solution formula obtained by Ukai [13]. The solution is represented by the Gauss kernel and various Riesz operators. It is known by Carpio [4] that the solution $u = G_t * u_0$ of the heat equation with initial data $u_0 \in L^1(\mathbb{R}^n)$ enjoys

$$\|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_1 t^{-1/2}\|u_0\|_1 \tag{5}$$

If the solution of (1) were represented only by G_t and a Riesz operator in $\mathbb{R}^n$, (6) could yield (4) since the Riesz operator is bounded in $\mathcal{H}^1$. Unfortunately, the formula contains the Riesz operator in tangential variables $x' = (x_1, \ldots, x_{n-1})$ to $\partial\mathbb{R}^n_+$, it is not clear that such operators are bounded in $\mathcal{H}^1(\mathbb{R}^n)$. To overcome this difficulty, we rewrite Ukai's formula so that ∇u does not have tangential Riesz operators with use of the operator Λ whose symbol equals $|\xi'|$, where $(\xi', \xi_n) = \xi \in \mathbb{R}^n$. Because of this, we need to prove

$$\|\Lambda u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_2 t^{-1/2}\|u_0\|_1 \tag{6}$$

in addition to (5). Although there are several extra technical difficulty, because of the formula, this is a rough idea for the proof of (4).

References

1. W. Borchers and T. Miyakaya, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. **282** (1988), 139–155.
2. ———, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189–227.
3. ———, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains, II*, Hiroshima Math. J. **21** (1991), 621–640.
4. A.Carpio, *Large time behavior in incompressible Navier-Stokes equations*, SIAM J. Math. Anal. **27** (1996), 449-475.
5. Z.-M.Chen, *Solution of the stationary and nonstationary Navier-Stokes equations in exterior domains*, Pacific J. Math. **159** (1993), 227–240.
6. C.Fefferman and E.Stein, *$\mathcal{H}^p$ spaces of several variavles*, Acta Math. **129** (1972), 137–197.
7. Y.Giga, *Analyticity of the semigroup generated by the Stokes operator in L^r spaces*, Math. Z. **178** (1981), 297–329.
8. Y.Giga and H.Sohr, *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **36** (1989), 103–130.
9. Y.Giga and H.Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. of Functional Analysis **102** (1991), 72–94.
10. H.Iwashita, *L_q-L_r estimate for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces*, Math. Ann. **285** (1989), 265–288.
11. T.Miyakawa, *Hardy spaces of solenoidal vector fields, with application to the Navier-Stokes equations*, Kyushu J. Math. **50** (1997), 1–64.
12. A.Torchinsky, *Real-variable methods in harmonic anarysis*, Academic Press, 1986.
13. S.Ukai, *A solution formula for the Stokes equation in $\mathbb{R}^n_+$*, Comm. Pure Appl. Math. **XL** (1987), 611–621.